# MATH 210: Introduction to Analysis

## Fall 2017-2018, Midterm 2, Duration: 60 min.

Name: Solution

| Exercise | Points | Scores |
|---|---|---|
| 1 | 40 | |
| 2 | 10 | |
| 3 | 15 | |
| 4 | 20 | |
| 5 | 25 | |
| Total | 110 | |

**INSTRUCTIONS:**

(a) Explain your answers precisely and clearly to ensure full credit.

(b) No book. No notes. No calculators.

**CHEAT SHEET:**

(a) Let $(X, d)$ be a metric space. A function $f : X \to \mathbb{R}$ is *Lipschitz* if there exists a constant $C > 0$ such that for all $x, y \in X$ we have

$$|f(x) - f(y)| \leq Cd(x, y).$$

(b) Recall that $d_\infty$ is defined on $\mathbb{R}^2$ as

$$d_\infty(x, y) := \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

for $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ is onto if for every $y \in \mathbb{R}$ there is $x \in \mathbb{R}$ such that $f(x) = y$.

**Exercise 1.** Warning: (a), (b) and (c) are independent.

(a) **(5 points)** Let $(X, d)$ be a metric space. State the definition of convergence of a sequence $\{x_n\}$ of elements of $X$.

See below

HW (b) **(10 points)** Prove that a sequence $\{(a_n, b_n)\}$ converges to $(a, b) \in \mathbb{R}^2$ for $d_\infty$ if and only if $\{a_n\}$ and $\{b_n\}$ respectively converge to $a$ and $b$.

$\Rightarrow$] Let $\varepsilon > 0$. There is $n_0$ st for $n \geq n_0$

$d_\infty((a_n, b_n), (a, b)) < \varepsilon$, that is

$\max(|a_n - a|, |b_n - b|) < \varepsilon$, that is

$|a_n - a| < \varepsilon$ and $|b_n - b| < \varepsilon$.

$\Leftarrow$] Let $\varepsilon > 0$.

There is $n_1$ st for $n \geq n_1$ $|a_n - a| < \varepsilon$

There is $n_2$ st for $n \geq n_2$ $|b_n - b| < \varepsilon$

So for $n \geq \max(n_1, n_2)$:

$|a_n - a| < \varepsilon$ and $|b_n - b| < \varepsilon$

that is $\max\{|a_n - a|, |b_n - b|\} < \varepsilon$

that is $d_\infty((a_n, b_n), (a, b)) < \varepsilon$.

(c) Let $(X, d)$ be a metric space. Let $K$ be a nonempty compact subset of $X$ and let $x_0 \in X$. Define

$$d(x_0, K) = \inf\{d(x_0, y) \mid y \in K\}.$$

HW

i. **(5 points)** Show that the function $x \mapsto d(x_0, x)$ defined on $X$ is Lipschitz.

$d(x_0, x) \leq d(x_0, y) + d(y, x)$

$\Rightarrow d(x_0, x) - d(x_0, y) \leq d(y, x)$

Similarly $d(x_0, y) - d(x_0, x) \leq d(y, x)$

$\Rightarrow |d(x_0, y) - d(x_0, x)| \leq d(y, x)$

(2nd Triangular inequality).

HW

ii. Suppose now that $X = \mathbb{R}^2$.

1. **(5 points)** Show that there exists $y \in K$ such that $d(x, y) = d(x, K)$.

$d(x_0, \cdot) : K \to \mathbb{R}$ is $C^0$ on compact Because Lip by i.

$\Rightarrow$ reaches it means, i.e there is $y$ s.t

$d(x_0, K) = d(x_0, y)$

HW

2. **(5 points)** Is $y$ necessarily unique?

No take $K = \{(1,0), (-1,0)\}$

$x_0 = (0,0)$

$1 = d(x_0, K) = d(x_0, (1,0)) = d(x_0, (-1,0))$

3. (**5 points**) Prove that a point $x \in X$ belongs to $K$ if and only if $d(x, K) = 0$.

$\Rightarrow$ $0 = d(x,x) \geq d(x,K)$

So $d(x,K) = 0$

$\Leftarrow$ By ii. 1, there is $y \in K$ s.t

$0 = d(x,K) = d(x,y)$

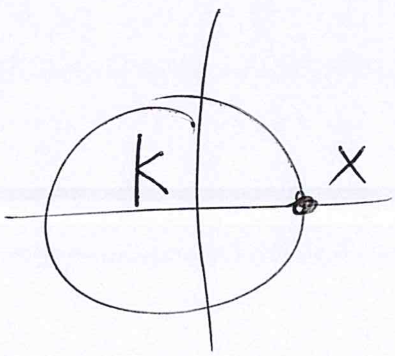
So $x = y$ (Property of metric).

So $x \in K$.

4. (**5 points**) Would 3. still hold if $K$ is not assumed to be compact?

No, take $K = B(0,1)$ (open ball)

$x = (1,0)$



**Exercise 2.**

(a) **(5 points)** Let $\sum a_n$ be a series of positive terms. Assume that $\overline{\lim}(a_n)^{\frac{1}{n}} > 1$. Use a property of $\overline{\lim}$ to show that $\sum a_n$ diverges.

$\overline{\lim}\, a_n^{\frac{1}{n}} > 1 \Rightarrow$ There are $\infty$ many $a_n$'s with $a_n^{\frac{1}{n}} > 1$

$\Rightarrow a_n \not\to 0$

$\Rightarrow \sum a_n$ div.

HW (b) **(5 points)** Use the previous question to prove that $\sum a_n$ with $a_n = \left(2 + \sin\frac{n\pi}{4}\right)^n r^n$ diverges if $r > \frac{1}{3}$.

Note $a_n^{\frac{1}{n}}$ is either $2r$, $(2+\sqrt{2})r$, $3r$ or $(2-\sqrt{2})r$

It follows that

$\overline{\lim}\, a_n^{\frac{1}{n}} = 3r$

By (a) $3r > 1 \Rightarrow \sum a_n$ diverges.

**Exercise 3.**

(a) Let $\sum a_n$ be an absolutely convergent series and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function.

i. **(5 points)** Assume that $f(0) = 0$. Prove that $\sum f(a_n)$ is absolutely convergent.

$$|f(a_n)| = |f(a_n) - f(0)| \leq C|a_n|$$

DCT $\Rightarrow$ $\sum f(a_n)$ cv absolutely.

ii. **(5 points)** Assume that $f(0) \neq 0$. Prove or disprove, using an explicit counterexample that $\sum f(a_n)$ is convergent.

Counterexample: Take $f \equiv 1$, $\sum a_n = \sum \frac{1}{n^2}$

(b) **(5 points)** Find a convergent series $\sum a_n$ and a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that $\sum f(a_n)$ is divergent.

Take $f(x) = |x|$, $\sum a_n = \sum \frac{(-1)^n}{n}$

Note $f$ is Lip by 2nd Triangle ineq.

$$||x| - |y|| \leq |x - y|.$$

**Exercise 4.** Consider the set

$$A = \left\{ (x,y) \in \mathbb{R}^2 \mid y \geq \frac{1}{x^2} \right\}$$

(a) **(15 points)** Show using two different methods that the set $A$ is closed.

Method 1: $A = f^{-1}(\underbrace{[1,\infty)}_{\text{close}})$ with $f : \mathbb{R}\times\mathbb{R} \xrightarrow{C^0} \mathbb{R}$, $(x,y) \mapsto x^2 y$

Method 2 (Sequence)

Let $\underbrace{(x_n,y_n)}_{\in A} \longrightarrow (x,y) \in \mathbb{R}^2$.

We have $y_n \geq \frac{1}{x_n^2}$

$\downarrow$

$y \geq \frac{1}{x^2}$

$\Rightarrow (x,y) \in A$

(b) **(5 points)** Prove or disprove that $A$ is compact.

$A$ is not bdd so not compact.

Indeed, $\underbrace{\{(n,1)\}}_{\text{the sequence}} \in A$ but not contained into a ball.

**Exercise 5.**

(a) **(5 points)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$. Prove that $f$ is onto.

HW

~~Let~~ Let $y \in \mathbb{R}$.

For $x$ large enough $f(x) > y$ (Since $\lim_{x\to+\infty} f(x) = +\infty$)

For $x$ large enough but $< 0$ $f(x) < y$ (Since $\lim_{x\to-\infty} f(x) = -\infty$)

By IVT there is $x$ s.t. $f(x) = y$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = +\infty$.

i. **(5 points)** Show that there is $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$.

By Contrad° if $f(x) \leq 0$ for all $x \in \mathbb{R}$

Then $\lim_{x\to\pm\infty} f(x)$ can't be $+\infty$.

$\Rightarrow$ There is $x_0 \in \mathbb{R}$ st $f(x_0) > 0$.

ii. **(5 points)** Show that there is $R > 0$ such that $f(x) > f(x_0)$ if $|x| > R$.

~~For $f(x_0) >$~~

By def°, there is $R_1 > 0$ st for $x > R_1$

we have $f(x) > f(x_0)$.

there is $R_2 > 0$ st for $x < -R_2$

we have $f(x) > f(x_0)$.

$\Rightarrow$ for $|x| > R = \max\{R_1, R_2\}$, $f(x) > f(x_0)$.

iii. **(5 points)** Show that there is $x_1$ such that $f(x) \geq f(x_1)$ if $|x| \leq R$.

$f$ is $C^0$ on $[-R, R]$ compact so reaches its min, that is there is $x_1$ s.t

$$f(x) \geq f(x_1) \text{ on } [-R, R].$$

iv. **(5 points)** Deduce that $f$ admits a global minimum, namely that there is a point $x^*$ such that $f(x) \geq f(x^*)$ for all $x \in \mathbb{R}$.

Set $f(x^*) = \min\{f(x_0), f(x_1)\}$

(i.e $x^* = x_0$ if $f(x_1) \geq f(x_0)$
and $x^* = x_1$ if $f(x_0) \geq f(x_1)$)

By ii & iii we have

$$f(x) \geq f(x^*) \text{ on } \mathbb{R}.$$